

Theory of Čerenkov and transition radiation from layered structures

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A scattering theoretical approach is used to describe the generation of both transition and Čerenkov radiation due to the passage of fast electrons through layered, in general absorptive, dielectrics. It leads to a considerable improvement over the coherent summation method usually employed. Reflection of the produced radiation from the layer interfaces is now properly taken into account and absorption of radiation, if present, is automatically taken care of. The usual restriction that the energy of the produced photons must be small relative to the initial electron energy is lifted. In contrast to existing theories the production of Čerenkov radiation, if it takes place, is included as well. Our expressions for radiation production feature the eigenmodes of the Helmholtz equation for the dielectric and we discuss how the latter can be obtained by means of a transfer matrix formalism. A numerical evaluation shows that under the appropriate conditions our results are in agreement with those from the coherent summation formalism. Finally, we present numerical results that give an impression of the relative yields for transition and Čerenkov radiation.

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I. INTRODUCTION

A. Background

Fast electrons moving through a dielectric medium can produce radiation through a number of different mechanisms. Well known is the Čerenkov radiation (CR), which is produced throughout the medium provided the electron velocity exceeds the phase velocity of the radiation inside the medium. A second type of radiation is transition radiation (TR). Here the electron energy can be arbitrary but the medium must be spatially inhomogeneous, i.e., the electric permeability $\epsilon(\mathbf{x})$ must depend on \mathbf{x} . This leads to a variable charge density

$$\rho(\mathbf{x}) = \partial_{\mathbf{x}} \cdot \mathbf{E}(\mathbf{x}) = [\partial_{\mathbf{x}} \epsilon^{-1}(\mathbf{x})] \cdot \mathbf{D}(\mathbf{x}), \quad (1.1)$$

causing the electron to accelerate and decelerate with the accompanying production of radiation. Thus it can be viewed as a type of bremsstrahlung (BS) associated with the polarization properties of the dielectric. At a sharp interface, TR over a wide frequency range can be generated.

Third, direct BS, due to collisions of the electron with the atoms or ions constituting the dielectric can also occur. On the other hand, such collisions deflect the electron and they act as a loss mechanism for TR and CR production. This is the case in particular in a periodic layered medium with normally incident electrons, where the periodicity is used to enhance TR production at a given frequency ω .

Throughout the years there has been some interest in the possibility of using TR and CR generated in this way as a simple source for x-ray radiation. For recent experimental results, see [1]. Since the TR produced at an interface is proportional to $(\Delta\epsilon)^2$, $\Delta\epsilon$ being the jump in ϵ across it,

layers with large differences in ϵ are required for efficient radiation production. In particular, in Ref. [2], where an extensive list of earlier references is given as well, Kaplan *et al.* explore theoretically the use of the system of interfaces present in a multilayer in combination with MeV electrons for TR production. Here absorptive dielectrics at frequencies near absorption edges for inner shell excitation are interesting candidates, since they lead to extremely large contrasts in $\epsilon(\omega)$ and correspondingly increased yields. These authors also show that around such absorption edges direct BS can be neglected. The above structures are also interesting for CR production, since it is precisely near the absorption edges where the Čerenkov radiation condition can be met. The idea of using dispersive dielectrics for soft x-ray production was considered earlier by Bazylev *et al.* [3] in connection with CR.

B. Existing theory

The usual theoretical setup [4] for a description of TR starts off from the radiation produced at a single interface. The total radiation emanating from a multilayer is then obtained by coherently summing the contributions from the individual interfaces. In addition, it is common practice to assume that the energy of the produced photons E_{ph} is small relative to the initial electron energy E_{el} . This allows the electron velocity to be taken as constant, in which case the radiation produced is obtained from Maxwell's equations with a prescribed external current density. This makes sense in technological applications, where the use of MeV electrons is considered for the generation of photons with energies of a few hundred eV or lower. In [2] further amendments are made to include the effects of photon absorption in an absorptive dielectric and the scattering of electrons from the individual atoms or ions constituting the material (electron-electron and electron-plasmon scattering are discarded as being negligible at high electron energies). Photon absorption is included through an exponential damping factor and electron scattering is treated as an attenuation effect.

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A less elegant feature of this approach is the apparent need to treat the outer layers of a stack separately. An important drawback inherent to the method is that CR, being a bulk effect, is not simultaneously taken into account. In addition, systems with gradually changing ε can only be treated perturbatively.

C. Present approach

A basic flaw of the usual theory is that CR is not included. This makes it difficult to make a choice between TR and CR as the basic process for x-ray production. However, the description of TR and CR can be unified by realizing that we are essentially dealing with a scattering phenomenon, $e_- + \text{vacuum} \rightarrow e_- + \text{photons}$. As is known from quantum electrodynamics (QED), such processes lead to expressions depending on the field modes. In vacuum QED the latter are simple plane waves but in macroscopic media they become the eigenmodes (HE's) of the Helmholtz operator, the latter now featuring a nontrivial $\varepsilon(\mathbf{x})$. Much activity is taking place in connection with the decay of excited atoms embedded in an optical material such as a photonic crystal, $\text{atom}^* + \text{vacuum} \rightarrow \text{atom} + \text{photon}$. There the decay of the atom is described in terms of the so-called local density of states, which in turn can be expressed in terms of the HE's (see [5] and references quoted there). As far as we are aware, the only paper in which the HE's make their appearance in connection with TR is one by Glauber and Lewenstein [6], where the quantization of a linear nonabsorptive dielectric is discussed. As an application, the generation of TR is considered for the case of a quantized electromagnetic field coupled to a classical current density. After specializing to the single interface case the relevant HE's are determined and the standard Ginzburg-Frank formula [4] for a single interface is recovered under the appropriate conditions.

Below we present a general approach toward TR and CR in the same spirit. We make use of results recently obtained by one of us (Tip [7]), where both classical and quantized linear absorptive dielectrics are considered (referred to as LAD in the following). We consider two cases involving layered dielectrics with finite width.

In Sec. II we study a classical absorptive dielectric with a given external current density, i.e., the familiar situation with $E_{ph} \ll E_{el}$. However, our medium is a general absorptive one, and, since we avoid the coherent summation procedure, CR is also included and our results apply as well to media with gradually varying $\varepsilon(\mathbf{x})$. The final result involves the momentum space version (Fourier transform) of the HE's.

We then turn to the fully quantized situation in Sec. III. Since usually electron beams with high energy are considered, we describe the electron through the relativistic Schrödinger equation (spin effects can safely be discarded, the energies involved in spin-flip and similar processes being extremely small relative to E_{el} and E_{ph}). Since multiphoton production is negligible as compared to single photon generation, a calculation to leading order in the coupling (fine structure) constant suffices. This case was already considered in LAD. Here we simply give the result obtained there. It is interesting to note that the final result does not involve \hbar ; it is essentially classical. In addition, no assumption about the relative magnitudes of E_{el} and E_{ph} is required, so highly

inelastic processes are covered as well. However, if $E_{ph} \ll E_{el}$, the result of Sec. II is recovered.

The final step involves the evaluation of the HE's. We take this up in Sec. IV, where we start by obtaining Lippmann-Schwinger and eigenvalue equations. The translation invariance in directions parallel to the interfaces leads to the usual decomposition into two independent problems for the two different polarizations. For each of these a transfer matrix formalism can then be set up. In the special case of a layered structure with piecewise constant $\varepsilon(\mathbf{x})$, the transfer matrix simply becomes a product of individual matrices, each pertaining to a specific layer. This fixes the HE's in coordinate representation and a numerical Fourier transformation can then be executed to obtain results for specific cases. In Sec. V, the discussion section, we give a few examples and comparisons with results obtained through the coherent summation method. Under the appropriate conditions the agreement is reasonable.

D. Some notions from scattering theory

As said, we shall make use of scattering theoretical concepts, in particular of the Møller or wave operators Ω_{\pm} . The latter appear in a natural way if we note that in a scattering situation [$H = H_0 + V$ is the full Hamiltonian, H_0 the free one, and $\psi(t)$ the state in the Schrödinger case] the actual motion $\psi(t) = \exp(-iHt)\psi(0)$ approaches the free motion $\exp(-iH_0t)\varphi$, with φ an appropriate state vector, as $t \rightarrow \pm\infty$, so

$$\Omega_{\pm}\psi(0) = \lim_{t \rightarrow \pm\infty} \exp(iHt)\exp(-iH_0t)\varphi \quad (1.2)$$

exist. Then $S = \Omega_{+}^{*}\Omega_{-}$ is the scattering operator for the process considered. We shall also make use of the feature that if $\varphi_{\lambda}^{(0)}$ is an eigenvector of H_0 at the eigenvalue λ , then $\varphi_{\lambda} = \Omega_{\pm}\varphi_{\lambda}^{(0)}$ is an eigenvector of H at the same eigenvalue,

$$H_0\varphi_{\lambda}^{(0)} = \lambda\varphi_{\lambda}^{(0)} \Rightarrow H\varphi_{\lambda} = \lambda\varphi_{\lambda}. \quad (1.3)$$

There exists a vast, mainly mathematical-physical literature about the existence and completeness (i.e., the unitarity of S) of wave operators, in particular for the Schrödinger case. Some useful references are [8–10]. Applications to electromagnetic scattering can be found in [7] and [11].

Concerning notation we note the following: The vacuum electric and magnetic permeabilities are denoted by ε_0 and μ_0 , respectively. Thus the speed of light in vacuum is $c_0 = (\varepsilon_0\mu_0)^{-1/2}$. The region of space where the medium is non-absorptive (conservative) is indicated as \mathcal{M}_{na} , whereas the absorptive region is denoted as \mathcal{M}_a . When considering layered media we shall always take the interfaces parallel to the X_1 - X_2 plane, \mathbf{e}_j will be the unit vector in the X_j direction, and \mathbf{a}^{\perp} refers to the component of the vector \mathbf{a} perpendicular to the X_3 direction.

II. RADIATION PRODUCED BY A GIVEN CURRENT

A. Maxwell's equations

Our starting point is the set of Maxwell's equations for a linear, in general absorptive, dielectric in the presence of an external current density $\mathbf{J}(\mathbf{x}, t)$,

$$\partial_t \mathbf{D}(\mathbf{x}, t) = \partial_{\mathbf{x}} \times \mathbf{H}(\mathbf{x}, t) - \mathbf{J}(\mathbf{x}, t), \quad (2.1)$$

$$\partial_t \mathbf{B}(\mathbf{x}, t) = -\partial_{\mathbf{x}} \times \mathbf{E}(\mathbf{x}, t), \quad \partial_{\mathbf{x}} \cdot \mathbf{B}(\mathbf{x}, t_0) = 0.$$

We assume that magnetization is absent, so $\mathbf{B} = \mu_0^{-1} \mathbf{H}$, and

$$\mathbf{D}(\mathbf{x}, t) = \begin{cases} \varepsilon_1(\mathbf{x}) \mathbf{E}(\mathbf{x}, t) = \varepsilon_0 (1 + \chi_{stat}) \mathbf{E}(\mathbf{x}, t), & \mathbf{x} \in \mathcal{M}_{na} \\ \varepsilon_0 \left[\mathbf{E}(\mathbf{x}, t) + \int_{t_0}^t ds \chi(\mathbf{x}, t-s) \mathbf{E}(\mathbf{x}, s) \right], & \mathbf{x} \in \mathcal{M}_a. \end{cases} \quad (2.2)$$

Here t_0 is some initial time, taken to be $-\infty$ later on, and $\chi(\mathbf{x}, t)$ is the electric susceptibility. Since the latter vanishes outside \mathcal{M}_a we can write, setting $\varepsilon_1(\mathbf{x}) = \varepsilon_0$ for $\mathbf{x} \in \mathcal{M}_a$, so $\varepsilon_1(\mathbf{x}) \chi(\mathbf{x}, t) = \varepsilon_0 \chi(\mathbf{x}, t)$,

$$\mathbf{D}(\mathbf{x}, t) = \varepsilon_1(\mathbf{x}) \left(\mathbf{E}(\mathbf{x}, t) + \int_{t_0}^t ds \chi(\mathbf{x}, t-s) \mathbf{E}(\mathbf{x}, s) \right). \quad (2.3)$$

We recall that the nonabsorptive or static situation follows if the kernel $\chi(t)$ is decaying rapidly. Then

$$\begin{aligned} \mathbf{D}(\mathbf{x}, t) &= \varepsilon_0 \left(\mathbf{E}(\mathbf{x}, t) + \int_0^{t-t_0} ds \chi(\mathbf{x}, s) \mathbf{E}(\mathbf{x}, t-s) \right) \\ &\approx \varepsilon_0 \left(1 + \int_0^{t-t_0} ds \chi(\mathbf{x}, s) \right) \mathbf{E}(\mathbf{x}, t) \\ &\approx \varepsilon_0 \left(1 + \int_0^{\infty} ds \chi(\mathbf{x}, s) \right) \mathbf{E}(\mathbf{x}, t) \\ &= \varepsilon_0 [1 + \chi_{stat}(\mathbf{x})] \mathbf{E}(\mathbf{x}, t). \end{aligned} \quad (2.4)$$

In LAD a general approach toward linear absorptive dielectrics was presented. There, by means of introducing two auxiliary fields, it was shown that energy conservation can be restored for the combined set of fields, and that this set is the

solution of a set of coupled first order equations and is without time convolution terms. Here we briefly recapitulate the procedure. We note that in general $\chi(\mathbf{x}, 0) = 0$ and that $\chi'(\mathbf{x}, t) = \partial_t \chi(\mathbf{x}, t)$, extended to negative t according to $\chi'(\mathbf{x}, -t) = \chi'(\mathbf{x}, t)$, has a non-negative Fourier transform, so

$$\chi'(\mathbf{x}, t) = \int_{-\infty}^{+\infty} d\omega \exp(-i\omega t) \sigma(\mathbf{x}, \omega)^2, \quad \sigma(\mathbf{x}, \omega) \geq 0. \quad (2.5)$$

Introducing Eq. (2.4) into Eq. (2.1) we have

$$\begin{aligned} \partial_t \varepsilon_1(\mathbf{x}) \mathbf{E}(\mathbf{x}, t) &= \partial_{\mathbf{x}} \times \mu_0^{-1} \mathbf{B}(\mathbf{x}, t) \\ &\quad - \varepsilon_0 \int_{t_0}^t ds \chi'(\mathbf{x}, t-s) \mathbf{E}(\mathbf{x}, s) - \mathbf{J}(\mathbf{x}, t). \end{aligned} \quad (2.6)$$

Then, with $\mathbf{F}_1(\mathbf{x}, t) = \varepsilon_1(\mathbf{x})^{1/2} \mathbf{E}(\mathbf{x}, t)$, $\mathbf{F}_3(\mathbf{x}, t) = \mu_0^{-1/2} \mathbf{B}(\mathbf{x}, t)$, and with two new auxiliary real vector fields $\mathbf{F}_2(\mathbf{x}, \omega, t)$ and $\mathbf{F}_4(\mathbf{x}, \omega, t)$, which vanish for $t = t_0$, $\mathbf{F}_2(\mathbf{x}, \omega, t_0) = \mathbf{F}_4(\mathbf{x}, \omega, t_0) = 0$,

$$\begin{aligned} \partial_t \mathbf{F}_1(\mathbf{x}, t) &= \varepsilon_1(\mathbf{x})^{-1/2} \partial_{\mathbf{x}} \times \mu_0^{-1/2} \mathbf{F}_3(\mathbf{x}, t) \\ &\quad + \int d\omega \sigma(\mathbf{x}, \omega) \mathbf{F}_4(\mathbf{x}, \omega, t) - \varepsilon_1(\mathbf{x})^{-1/2} \mathbf{J}(\mathbf{x}, t), \\ \partial_t \mathbf{F}_2(\mathbf{x}, \omega, t) &= \omega \mathbf{F}_4(\mathbf{x}, \omega, t), \\ \partial_t \mathbf{F}_3(\mathbf{x}, t) &= -\mu_0^{-1/2} \partial_{\mathbf{x}} \times \varepsilon_1(\mathbf{x})^{-1/2} \mathbf{F}_1(\mathbf{x}, t), \\ \partial_t \mathbf{F}_4(\mathbf{x}, \omega, t) &= -\omega \mathbf{F}_2(\mathbf{x}, \omega, t) - \sigma(\mathbf{x}, \omega) \mathbf{F}_1(\mathbf{x}, t). \end{aligned} \quad (2.7)$$

From this set Maxwell's equations are retrieved by expressing the auxiliary fields \mathbf{F}_2 and \mathbf{F}_4 in terms of the electromagnetic ones and substituting them in the remaining equations. In compact notation, with

$$\mathbf{F}(t) = \begin{pmatrix} \mathbf{F}_1(t) \\ \mathbf{F}_2(t) \\ \mathbf{F}_3(t) \\ \mathbf{F}_4(t) \end{pmatrix}, \quad \mathbf{G}(t) = \begin{pmatrix} \varepsilon_1^{-1/2} \mathbf{J}(t) \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{N} = \begin{pmatrix} 0 & 0 & (\varepsilon_1 \mu_0)^{-1/2} \partial_{\mathbf{x}} \times \int d\omega \sigma(\omega) \dots \\ 0 & 0 & 0 & \omega \\ -\partial_{\mathbf{x}} \times (\varepsilon_1 \mu_0)^{-1/2} & 0 & 0 & 0 \\ -\sigma(\omega) & -\omega & 0 & 0 \end{pmatrix}, \quad (2.8)$$

we have

$$\partial_t \mathbf{F}(t) = \mathbf{N} \mathbf{F}(t) - \mathbf{G}(t) = -i \mathbf{K} \mathbf{F}(t) - \mathbf{G}(t). \quad (2.9)$$

Noting that the auxiliary fields vanish outside the absorptive region \mathcal{M}_a , the conserved energy is now

$$\begin{aligned} \mathcal{E} &= \frac{1}{2} \langle \mathbf{F} | \mathbf{F} \rangle = \frac{1}{2} \int d\mathbf{x} [\mathbf{F}_1(\mathbf{x})^2 + \mathbf{F}_3(\mathbf{x})^2] \\ &\quad + \int_{\mathcal{M}_a} d\mathbf{x} \int d\omega [\mathbf{F}_2(\mathbf{x}, \omega)^2 + \mathbf{F}_4(\mathbf{x}, \omega)^2]. \end{aligned} \quad (2.10)$$

If we suppose that initially, at $t = t_0$, the charged particles

producing the current density have not yet reached the medium, all initial fields vanish and Eq. (2.8) has the solution

$$\begin{aligned} \mathbf{F}(t) &= - \int_{t_0}^t ds \exp[-i\mathbf{K}(t-s)]\mathbf{G}(s) \\ &\xrightarrow{t_0 \rightarrow -\infty} - \int_{-\infty}^t ds \exp[-i\mathbf{K}(t-s)]\mathbf{G}(s) \\ &= - \int_0^\infty ds \exp[-i\mathbf{K}(s)]\mathbf{G}(t-s). \end{aligned} \quad (2.11)$$

B. Radiated energy in terms of wave operators

Let \mathbf{K}_0 be the generator for the free time evolution, i.e., $\varepsilon = \varepsilon_0$ and $\chi = 0$ (note that electromagnetic and auxiliary fields are not coupled by \mathbf{K}_0). In case the dielectric is finite or a layered structure with finite width, we can introduce the Møller or wave operator

$$\Omega_+ = \lim_{t \rightarrow \infty} \exp(i\mathbf{K}t)\exp(-i\mathbf{K}_0t), \quad (2.12)$$

which is well defined when acting upon states \mathbf{F} with transverse electromagnetic components, in particular those for which the auxiliary components \mathbf{F}_2 and \mathbf{F}_4 vanish and the electromagnetic ones are plane wave eigenstates of \mathbf{K}_0 . More information about the use of wave operators in an electrodynamic context can be found in LAD and [11]. Now

$$\begin{aligned} \exp(i\mathbf{K}_0t)\mathbf{F}(t) &= - \exp(i\mathbf{K}_0t)\exp(i\mathbf{K}t) \\ &\quad \times \int_{-\infty}^t ds \exp(i\mathbf{K}s)\mathbf{G}(s) \\ &\xrightarrow{t \rightarrow \infty} - \Omega_+^* \int_{-\infty}^t ds \exp(i\mathbf{K}s)\mathbf{G}(s) \\ &\xrightarrow{t \rightarrow \infty} - \Omega_+^* \int_{-\infty}^\infty ds \exp(i\mathbf{K}s)\mathbf{G}(s) \\ &\approx - \int_{-\infty}^\infty ds \exp(i\mathbf{K}_0s)\Omega_+^* \mathbf{G}(s), \end{aligned} \quad (2.13)$$

so

$$\begin{aligned} \mathbf{F}(t) &\xrightarrow{t \rightarrow \infty} - \exp(-i\mathbf{K}_0t) \int_{-\infty}^\infty ds \exp(i\mathbf{K}_0s)\Omega_+^* \mathbf{G}(s) \\ &= - \exp(-i\mathbf{K}_0t)\mathbf{M}, \end{aligned} \quad (2.14)$$

where

$$\mathbf{M} = \int_{-\infty}^\infty ds \exp(i\mathbf{K}_0s)\Omega_+^* \mathbf{G}(s). \quad (2.15)$$

Suppose the origin of our coordinate system is somewhere inside the dielectric. We calculate the amount of electromagnetic energy \mathcal{E}_C emerging in a cone \mathcal{C} in coordinate space,

centered in the origin and, in the layered case, with axis not parallel to the dielectric interfaces. In order to have no overlap with the dielectric, the cone is truncated for small x . Then, since the auxiliary fields vanish outside absorbing regions, the total energy contained in \mathcal{C} coincides with its electromagnetic part. Let $\chi_{\mathcal{A}}(f)$ be the characteristic function for the set \mathcal{A} , i.e., $\chi_{\mathcal{A}}(f) = 1$ for $f \in \mathcal{A}$ and vanishes otherwise. Then

$$\chi_C(\mathbf{x}) = \theta(\mathbf{e} \cdot \mathbf{e}_x - a)\theta(x - b), \quad (2.16)$$

where \mathbf{e} is the axis of the cone, a the cosine of its aperture, and b the truncation parameter. As discussed in LAD,

$$\begin{aligned} \lim_{t \rightarrow \infty} \exp(i\mathbf{K}_0t)\chi_C(\mathbf{x})\mathbf{P}_{em}\exp(-i\mathbf{K}_0t) \\ = [\theta(\mathbf{e} \cdot \mathbf{e}_p - a)\mathbf{P}_+ + \theta(-\mathbf{e} \cdot \mathbf{e}_p - a)\mathbf{P}_-]\mathbf{P}_{em}, \end{aligned} \quad (2.17)$$

where $\theta(\cdot)$ is the Heaviside step function, $\mathbf{p} = -i\partial_{\mathbf{x}}$, \mathbf{P}_\pm are the projectors upon the eigenspaces of \mathbf{K}_0 with positive and negative eigenvalues, respectively, and \mathbf{P}_{em} is the projector upon the electromagnetic components of \mathbf{F} . Now

$$\begin{aligned} \mathcal{E}_C(t) &= \frac{1}{2} \langle \mathbf{F}(t) | \chi_C(\mathbf{x}) \mathbf{P}_{em} | \mathbf{F}(t) \rangle \\ &\xrightarrow{t \rightarrow \infty} \frac{1}{2} \langle \mathbf{M} | \exp(i\mathbf{K}_0t)\chi_C(\mathbf{x})\mathbf{P}_{em}\exp(-i\mathbf{K}_0t) | \mathbf{M} \rangle \\ &\xrightarrow{t \rightarrow \infty} \approx \frac{1}{2} \langle \mathbf{M} | [\theta(\mathbf{e} \cdot \mathbf{e}_p - a)\mathbf{P}_+ \\ &\quad + \theta(-\mathbf{e} \cdot \mathbf{e}_p - a)\mathbf{P}_-] \mathbf{P}_{em} | \mathbf{M} \rangle = \mathcal{E}_C. \end{aligned} \quad (2.18)$$

Let \mathbf{C} be the conjugation operator,

$$(\mathbf{C}f)(\mathbf{x}) = \overline{f(\mathbf{x})}. \quad (2.19)$$

Then, since $\mathbf{N} = -i\mathbf{K}$ and $\mathbf{N}_0 = -i\mathbf{K}_0$ are real operators, $\mathbf{C}\mathbf{N}\mathbf{C} = \mathbf{N}$, $\mathbf{C}\mathbf{N}_0\mathbf{C} = \mathbf{N}_0$, so Ω_+ is invariant, $\mathbf{C}\Omega_+\mathbf{C} = \Omega_+$. Also $\mathbf{C}\mathbf{P}_-\mathbf{C} = \mathbf{P}_+$ and $\mathbf{C}\mathbf{p}\mathbf{C} = -\mathbf{p}$. Since \mathbf{G} is real, $\mathbf{C}\mathbf{G} = \mathbf{G}$ and it follows that the first and second terms in Eq. (2.18) are equal, so

$$\mathcal{E}_C = \langle \mathbf{M} | \theta(\mathbf{e} \cdot \mathbf{e}_p - a)\mathbf{P}_+\mathbf{P}_{em} | \mathbf{M} \rangle. \quad (2.20)$$

Next we note that the eigenvectors $\mathbf{w}_{\mathbf{k}j\alpha}^{(0)}$ of \mathbf{K}_0 ,

$$\mathbf{K}_0 \mathbf{w}_{\mathbf{k}j\alpha}^{(0)} = \alpha c_0 k \mathbf{w}_{\mathbf{k}j\alpha}^{(0)}, \quad (2.21)$$

for which the auxiliary components vanish, are given by

$$\mathbf{w}_{\mathbf{k}j\alpha}^{(0)}(\mathbf{x}) = \begin{pmatrix} \mathbf{n}_j \\ 0 \\ \mathbf{e}_k \times \mathbf{n}_j \\ 0 \end{pmatrix} \frac{\exp(i\alpha \mathbf{k} \cdot \mathbf{x})}{4\pi^{3/2}}, \quad \mathbf{k} \in \mathbb{R}^3, \quad j = 1, 2, \quad \alpha = \pm 1, \quad (2.22)$$

where $\mathbf{e}_k = \mathbf{k}/k$ and \mathbf{n}_1 and \mathbf{n}_2 are mutually orthonormal. The $\mathbf{w}_{\mathbf{k}j\alpha}^{(0)}$'s are normalized according to

$$\begin{aligned} \langle \mathbf{w}_{\mathbf{k}j\alpha}^{(0)} | \mathbf{w}_{\mathbf{k}'j'\alpha'}^{(0)} \rangle &= \int d\mathbf{x} \overline{\mathbf{w}_{\mathbf{k}j\alpha}^{(0)}(\mathbf{x})} \cdot \mathbf{w}_{\mathbf{k}'j'\alpha'}^{(0)}(\mathbf{x}) \\ &= \delta(\mathbf{k}-\mathbf{k}') \delta_{jj'} \delta_{\alpha\alpha'}. \end{aligned} \quad (2.23)$$

Since $K\Omega_+ = \Omega_+ K_0$, the corresponding eigenstates of K are

$$\mathbf{w}_{\mathbf{k}j\alpha} = \Omega_+ \mathbf{w}_{\mathbf{k}j\alpha}^{(0)}, \quad K\mathbf{w}_{\mathbf{k}j\alpha} = \alpha c_0 k \mathbf{w}_{\mathbf{k}j\alpha}. \quad (2.24)$$

Unless the medium is conservative, they have nonvanishing auxiliary components. We note that for finite media

$$\lim_{\mathbf{x} \rightarrow \infty} \mathbf{w}_{\mathbf{k}j\alpha}(\mathbf{x}) - \mathbf{w}_{\mathbf{k}j\alpha}^{(0)}(\mathbf{x}) \approx 0, \quad (2.25)$$

but the situation for layered media is different. We express $P_+ P_{em}$ in terms of the $\mathbf{w}_{\mathbf{k}j+}^{(0)}$'s:

$$P_+ P_{em} = \sum_j \int d\mathbf{k} |\mathbf{w}_{\mathbf{k}j+}^{(0)}\rangle \langle \mathbf{w}_{\mathbf{k}j+}^{(0)}|, \quad (2.26)$$

so, omitting the subscript $+$ from now on,

$$\begin{aligned} \mathcal{E}_C &= \sum_j \int d\mathbf{k} \theta(\mathbf{e} \cdot \mathbf{e}_{\mathbf{k}} - a) \langle M | \mathbf{w}_{\mathbf{k}j}^{(0)} \rangle \langle \mathbf{w}_{\mathbf{k}j}^{(0)} | M \rangle \\ &= \sum_j \int d\mathbf{k} \theta(\mathbf{e} \cdot \mathbf{e}_{\mathbf{k}} - a) \langle M | \mathbf{w}_{\mathbf{k}j}^{(0)} \rangle \langle \mathbf{w}_{\mathbf{k}j}^{(0)} | M \rangle \\ &= \sum_j \int_C d\mathbf{k} \langle M | \mathbf{w}_{\mathbf{k}j}^{(0)} \rangle \langle \mathbf{w}_{\mathbf{k}j}^{(0)} | M \rangle = \sum_j \int_C d\mathbf{k} |f_{\mathbf{k}j}|^2 \\ &= \sum_j \int_C d\mathbf{k} \mathfrak{E}_{\mathbf{k}j}, \end{aligned} \quad (2.27)$$

where $\mathfrak{E}_{\mathbf{k}j} d\mathbf{k}$ is the radiated energy with wave vector in $(\mathbf{k}, \mathbf{k} + d\mathbf{k})$ and polarization j and

$$\begin{aligned} f_{\mathbf{k}j} &= \langle M | \mathbf{w}_{\mathbf{k}j}^{(0)} \rangle \\ &= \int_{-\infty}^{\infty} dt \langle \exp(iK_0 s) \Omega_+^* G(t) | \mathbf{w}_{\mathbf{k}j}^{(0)} \rangle \\ &= \int_{-\infty}^{\infty} dt \langle G(t) | \Omega_+ \exp(-iK_0 s) \mathbf{w}_{\mathbf{k}j}^{(0)} \rangle \\ &= \int_{-\infty}^{\infty} dt \exp(-ikc_0 t) \langle G(s) | \mathbf{w}_{\mathbf{k}j} \rangle \\ &= \int_{-\infty}^{\infty} dt \int d\mathbf{x} \exp(-ikc_0 t) \varepsilon_1(\mathbf{x})^{-1/2} \mathbf{w}_{1\mathbf{k}j}(\mathbf{x}) \mathbf{J}(\mathbf{x}, t), \end{aligned} \quad (2.28)$$

where $\mathbf{w}_{1\mathbf{k}j}$ is the first (three-dimensional) component of $\mathbf{w}_{\mathbf{k}j}$. Thus we have expressed the emitted radiation in terms of the eigenvectors of K . Note that all radiation (i.e., both TR and CR) produced by $\mathbf{J}(\mathbf{x}, t)$ is included.

Next we specialize to the layered case, where the interfaces between the layers are parallel to the X_1 - X_2 plane. Thus we consider the situation that $\varepsilon_1(\mathbf{x}) = \varepsilon_1(x_3)$ and $\chi(\mathbf{x}, t) = \chi(x_3, t)$. The translation invariance in the X_1 and X_2 directions then gives

$$\varepsilon_1(x_3)^{-1/2} \mathbf{w}_{1\mathbf{k}j}(\mathbf{x}) = 2^{-1/2} \exp(i\mathbf{k}^\perp \cdot \mathbf{x}^\perp) \varphi_{\mathbf{k}j}(x_3), \quad (2.29)$$

(the factor $2^{-1/2}$ is introduced to ease a comparison with the quantum case, considered below), where \perp denotes the component of a vector orthogonal to the X_3 axis. Then, with

$$\mathbf{J}(\mathbf{x}, t) = e \mathbf{v} \delta(\mathbf{x} - \mathbf{v}t), \quad (2.30)$$

with e the electron charge and \mathbf{v} its velocity,

$$f_{\mathbf{k}j} = \frac{e}{\sqrt{2v_3}} \int dx_3 \exp[i(\mathbf{k}^\perp \cdot \mathbf{x}^\perp - kc_0)x_3/v_3] \varphi_{\mathbf{k}j}(x_3) \cdot \mathbf{v}, \quad (2.31)$$

and

$$\begin{aligned} \mathfrak{E}_{\mathbf{k}j} &= \frac{e^2}{2v_3^2} \left| \int dx_3 \exp[i(\mathbf{k}^\perp \cdot \mathbf{x}^\perp - kc_0)x_3/v_3] \varphi_{\mathbf{k}j}(x_3) \cdot \mathbf{v} \right|^2 \\ &= \frac{\pi e^2}{v_3^2} | \langle (\mathbf{k}^\perp \cdot \mathbf{x}^\perp - kc_0) \mathbf{v} / v_3 | \varphi_{\mathbf{k}j} \rangle |^2 \end{aligned} \quad (2.32)$$

in this case.

III. QUANTUM ELECTRODYNAMICAL APPROACH

An alternative to the formalism presented in the preceding section is to consider the radiation production process as a scattering phenomenon. Although this can be done on a purely classical basis, the corresponding quantum theory is much better developed and hence we shall use the latter. Thus the incoming state as $t \rightarrow -\infty$ is the product of an electron state and the vacuum state for the electromagnetic field,

$$\psi_{in} = \varphi_{el} \otimes \varphi_{vac}, \quad (3.1)$$

and the quantity we are interested in is the equivalent of \mathcal{E}_C above. Since multiphoton production is negligible relative to the single photon process, we make the restriction to the latter. This simplifies the formalism dramatically since it allows a first order perturbation calculation as discussed in LAD. Here we simply give the result. However, the procedure followed in LAD has a minor flaw, which is easily corrected. The point is that there we calculated the probability of finding a photon in the cone \mathcal{C} in coordinate space. This does not make sense since the electrodynamic parts of the vectors in the first Fock layer are transverse and $\chi_{\mathcal{C}}(\mathbf{x})$ acting upon the latter leads to longitudinal contributions as well. The situation is easily remedied by calculating instead the expected energy in \mathcal{C} , i.e., the expectation value, as time $t \rightarrow \infty$, of

$$\mathcal{E}_{\mathcal{C}} = \frac{1}{2} \int d\mathbf{x} \chi_{\mathcal{C}}(\mathbf{x}) [\mathbf{E}(\mathbf{x})^2 + \mathbf{B}(\mathbf{x})^2], \quad (3.2)$$

where \mathbf{E} and \mathbf{B} are now field operators. Here the auxiliary fields do not appear, since, as in the classical case, they vanish outside the absorptive region. The result obtained in LAD is still correct, except that here we also present the (minor) contribution from processes where the electron is reflected from the medium, a situation that is not included in the clas-

sical approach where the electron velocity is fixed. Thus we have for a layered medium with finite thickness,

$$\lim_{t \rightarrow \infty} \langle \mathcal{E}_C^{(1)}(t) \rangle = \sum_j \int_C d\mathbf{k} \mathfrak{E}_{\mathbf{k}j}, \quad \mathfrak{E}_{\mathbf{k}j} = \mathfrak{E}_{\mathbf{k}j}^{tr} + \mathfrak{E}_{\mathbf{k}j}^{ref}, \quad (3.3)$$

where the superscript (1) indicates that the contribution from the first Fock layer (single photon production) is taken, whereas the subscripts *tr* and *ref* refer to the cases where the electron is transmitted and reflected, respectively. Equation (3.3) is obtained in the case where the initial electron state is strongly peaked around the momentum \mathbf{k}_1 , $k_{13} > 0$, in the limit that it approaches a plane wave while keeping it normalized to unity (this approach would not work for a finite medium since there the overlap of the electron wave function and medium would tend to zero). Thus Eq. (3.3) gives the energy per electron. For its two components we have (a_{j3} is the component of \mathbf{a}_j along X_3)

$$\begin{aligned} \mathfrak{E}_{\mathbf{k}j}^{tr} &= \frac{e^2}{2k_{13}k_{23}} \left| \int dx_3 \exp[-i(k_{13}-k_{23})x_3] \varphi_{\mathbf{k}j}(x_3) \cdot \frac{1}{2}(\mathbf{k}_1 + \mathbf{k}_2) \right|^2 \\ &= \frac{e^2}{k_{13}k_{23}} \left\langle \frac{1}{2}(k_{23}-k_{13})(\mathbf{k}_1 + \mathbf{k}_2) \cdot \varphi_{\mathbf{k}j} \right\rangle^2, \\ \mathfrak{E}_{\mathbf{k}j}^{ref} &= \frac{e^2}{2k_{13}k_{23}} \left| \int dx_3 \exp[-i(k_{13}+k_{23})x_3] \varphi_{\mathbf{k}j}(x_3) \cdot \frac{1}{2}(\mathbf{k}_1 \right. \\ &\quad \left. + \mathbf{k}_2^\perp - k_{23}\mathbf{e}_3) \right|^2 \\ &= \frac{\pi e^2}{k_{13}k_{23}} \left\langle -\frac{1}{2}(k_{13}+k_{23})(\mathbf{k}_1 + \mathbf{k}_2^\perp - k_{23}\mathbf{e}_3) \cdot \varphi_{\mathbf{k}j} \right\rangle^2 \end{aligned} \quad (3.4)$$

where \mathbf{k}_2 with $k_{23} > 0$ is determined by the relations

$$\mathbf{k}_1^\perp = \mathbf{k}_2^\perp + \mathbf{k}^\perp, \quad \left[\mathbf{k}_1^\perp + \left(\frac{mc_0}{\hbar} \right)^{2/2} \right]^{1/2} = \left[\mathbf{k}_2^\perp + \left(\frac{mc_0}{\hbar} \right)^{2/2} \right]^{1/2} + k. \quad (3.5)$$

Since $\mathbf{p}_j = \hbar \mathbf{k}_j$ is the electron momentum, $\mathbf{p}_{ph} = \hbar \mathbf{k}$ the photon momentum, and the initial and final electron energies are $E_{in} = \hbar c_0 [\mathbf{k}_1^\perp + (mc_0/\hbar)^2]^{1/2}$, $E_{out} = \hbar c_0 [\mathbf{k}_2^\perp + (mc_0/\hbar)^2]^{1/2}$, whereas $E_{ph} = \hbar c_0 k$ is the photon energy, these relations express the conservation of the momentum components orthogonal to the X_3 axis and the energy. Finally, $\varphi_{\mathbf{k}j}(x_3)$ is the mode function introduced in Eq. (2.29).

We make the following remarks.

(a) Energy is conserved in the photon creation process. The absorption mechanism affects the created photons only when propagating through the medium and it enters the formalism through the mode functions $\varphi_{\mathbf{k}j}$.

(b) Since transition radiation is created only in space regions with steep gradients in the permeability, its production can be optimized by keeping absorptive layers thin, subject to coherence requirements.

(c) Although the above expressions were obtained through a quantum-electrodynamical calculation, they do not depend on \hbar and hence are in essence classical.

Let us now consider the situation we encountered earlier, i.e.,

$$E_{ph} \ll E_{in} \approx E_{out}. \quad (3.6)$$

It follows from the conservation laws that

$$\begin{aligned} k_{23} - k_{13} &= (k_{23} + k_{13})^{-1} \left\{ k_3^2 + 2\mathbf{k}_1^\perp \cdot \mathbf{k}^\perp \right. \\ &\quad \left. - 2k \left[\mathbf{k}_1^\perp + \left(\frac{mc_0}{\hbar} \right)^{2/2} \right]^{1/2} \right\} \\ &\approx \frac{\mathbf{k}_1^\perp \cdot \mathbf{k}^\perp - k [\mathbf{k}_1^\perp + (mc_0/\hbar)^2]^{1/2}}{k_{13}}. \end{aligned} \quad (3.7)$$

Using

$$\hbar \mathbf{k}_j = \mathbf{p}_j = [1 - (v_j/c_0)^2]^{-1/2} m \mathbf{v}_j, \quad (3.8)$$

and setting $\mathbf{v}_1 = \mathbf{v}$,

$$k_{23} - k_{13} \approx v_3^{-1} (\mathbf{k}^\perp \cdot \mathbf{v}^\perp - kc_0); \quad (3.9)$$

this leads to

$$\mathfrak{E}_{\mathbf{k}j}^{tr} \approx \frac{e^2}{2v_3^2} \left| \int dx_3 \exp[i(\mathbf{k}^\perp \cdot \mathbf{v}^\perp - kc_0)x_3/v_3] \varphi_{\mathbf{k}j}(x_3) \cdot \mathbf{v} \right|^2, \quad (3.10)$$

which is Eq. (2.32). Since $k_{23} + k_{13} \gg k_{23} - k_{13}$, the exponential in $\mathfrak{E}_{\mathbf{k}j}^{ref}$ is oscillating much faster than the one in $\mathfrak{E}_{\mathbf{k}j}^{tr}$, so $\mathfrak{E}_{\mathbf{k}j}^{ref} \ll \mathfrak{E}_{\mathbf{k}j}^{tr}$.

IV. MODE FUNCTIONS FOR A LAYERED MEDIUM

A. The Lippmann-Schwinger and eigenvalue equations

For a further evaluation of the expressions (2.32) and (3.4) we need the eigenmodes $\mathbf{w}_{\mathbf{k}j} = \Omega + \mathbf{w}_{\mathbf{k}j}^{(0)}$ where $\mathbf{w}_{\mathbf{k}j}^{(0)}$ is given by Eq. (2.21) with $\alpha = 1$. Some of the material discussed below can be found in LAD, but since there the vacuum permeabilities were set equal to 1, we briefly recapitulate a few matters. Our starting observation is that due to the skew-symmetric structure of \mathbf{K} , its square blocks out,

$$\mathbf{K}^2 = \begin{pmatrix} \mathbf{H}_e & 0 \\ 0 & \mathbf{H}_m \end{pmatrix}, \quad (4.1)$$

$$\mathbf{H}_e = \begin{pmatrix} c(\mathbf{x})\mathbf{H}_0 c(\mathbf{x}) & \int d\omega \omega \sigma(\mathbf{x}, \omega) \cdots \\ \omega \sigma(\mathbf{x}, \omega) & \omega^2 \end{pmatrix}.$$

Here $c(\mathbf{x}) = [\varepsilon_1(\mathbf{x})\mu_0]^{-1/2}$ and, with \mathbf{U} the unit 3×3 matrix, $\mathbf{H}_0 = \mathbf{p}^2 \mathbf{U} - \mathbf{p}\mathbf{p} = -\partial_x^2 \mathbf{U} + \partial_x \partial_x$. Setting

$$\mathbf{w}_{\mathbf{k}j} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{u}_{\mathbf{k}j} \\ \mathbf{v}_{\mathbf{k}j} \end{pmatrix} \quad (4.2)$$

and noting that

$$\mathbf{K}^2 \mathbf{w}_{\mathbf{k}j} = k^2 c_0^2 \mathbf{w}_{\mathbf{k}j}, \quad (4.3)$$

we have

$$\mathbf{H}_e \mathbf{u}_{\mathbf{k}j} = k^2 c_0^2 \mathbf{u}_{\mathbf{k}j}, \quad \mathbf{H}_m \mathbf{v}_{\mathbf{k}j} = k^2 c_0^2 \mathbf{v}_{\mathbf{k}j}. \quad (4.4)$$

The functions $\mathbf{u}_{\mathbf{k}j}$ and $\mathbf{u}_{\mathbf{k}j}^{(0)}$,

$$\mathbf{u}_{\mathbf{k}j}^{(0)}(\mathbf{x}) = \begin{pmatrix} (2\pi)^{-3/2} \mathbf{n}_j \exp(i\mathbf{k} \cdot \mathbf{x}) \\ 0 \end{pmatrix}, \quad (4.5)$$

are related by the wave operator (it differs from but is related to the one introduced in Sec. II)

$$\begin{aligned} \Omega &= \lim_{t \rightarrow \infty} \exp(i\mathbf{H}_e t) \exp(-i\mathbf{H}_e^{(0)} t) \\ &= 1 + i \lim_{\delta \rightarrow 0} \int_0^\infty dt \exp(-\delta t) \\ &\quad \times \exp(i\mathbf{H}_e t) \mathbf{V} \exp(-i\mathbf{H}_e^{(0)} t), \end{aligned} \quad (4.6)$$

i.e.,

$$\begin{aligned} \mathbf{u}_{\mathbf{k}j} &= \Omega \mathbf{u}_{\mathbf{k}j}^{(0)} \\ &= \mathbf{u}_{\mathbf{k}j}^{(0)} + i \lim_{\delta \rightarrow 0} \int_0^\infty dt \exp(-\delta t) \exp(i\mathbf{H}_e t) \mathbf{V} \\ &\quad \times \exp(-i\mathbf{H}_e^{(0)} t) \mathbf{u}_{\mathbf{k}j}^{(0)} \\ &= \mathbf{u}_{\mathbf{k}j}^{(0)} + (z^2 - \mathbf{H}_e)^{-1} \mathbf{V} \mathbf{u}_{\mathbf{k}j}^{(0)}, \end{aligned} \quad (4.7)$$

where $z^2 = k^2 c_0^2 - i\delta$ in the limit $\delta \downarrow 0$ and

$$\mathbf{H}_e = \mathbf{H}_e^{(0)} + \mathbf{V}, \quad \mathbf{H}_e^{(0)} = \begin{pmatrix} c_0^2 \mathbf{H}_0 & 0 \\ 0 & \omega^2 \end{pmatrix}. \quad (4.8)$$

In our expressions only the first component $\mathbf{u}_{1\mathbf{k}j}$ is needed. Taking the projection upon the first component of the right hand side of Eq. (4.7) and using the Feshbach formula (see LAD), we then arrive at the Lippmann-Schwinger equation

$$\begin{aligned} \mathbf{u}_1 &= \mathbf{u}_1^{(0)} + [z^2 - \mathbf{H}_{eff}(z^2)]^{-1} \mathbf{V}_{eff}(z^2) \mathbf{u}_1^{(0)} \\ &= [z^2 - \mathbf{H}_{eff}(z^2)]^{-1} [z^2 - c_0^2 \mathbf{H}_0] \mathbf{u}_1^{(0)}, \end{aligned} \quad (4.9)$$

where we dropped the subscript $\mathbf{k}j$ for brevity and

$$\begin{aligned} \mathbf{H}_{eff}(z^2) &= c \mathbf{H}_0 c - z^2 \hat{\chi}(z), \\ \mathbf{V}_{eff}(z^2) &= c \mathbf{H}_0 c - c_0^2 \mathbf{H}_0 - z^2 \hat{\chi}(z), \\ \hat{\chi}(z) &= \int_0^\infty dt \exp(izt) \chi(t), \quad z = (k^2 c_0^2 - i\delta) \xrightarrow{\delta \downarrow 0} -kc_0. \end{aligned} \quad (4.10)$$

Multiplying Eq. (4.9) by $z^2 - \mathbf{H}_{eff}(z^2)$ and in the limit $\delta \downarrow 0$ we obtain the eigenvalue equation

$$[k^2 c_0^2 - \mathbf{H}_{eff}(k^2 c_0^2 - i0)] \mathbf{u}_{1\mathbf{k}j} = 0. \quad (4.11)$$

B. The layered case

It is convenient to set

$$z^2 = \zeta^2 c_0^2, \quad \zeta^2 = k^2 - i\delta, \quad \zeta = -k + i\delta'. \quad (4.12)$$

Then

$$\begin{aligned} \varepsilon_1^{-1/2} \mathbf{u}_1 &= \varepsilon_1^{-1/2} \frac{c}{c_0} \left\{ \frac{z^2}{c^2} (1 + \hat{\chi}(z)) - \mathbf{H}_0 \right\}^{-1} \frac{c}{c_0} \left(\frac{z^2}{c_0^2} - \mathbf{H}_0 \right) \mathbf{u}_1^{(0)} \\ &= [\zeta^2 \varepsilon(\zeta) - \mathbf{H}_0]^{-1} \left(\frac{\varepsilon_1}{\varepsilon_0} \right)^{1/2} (\zeta^2 - \mathbf{H}_0) \mathbf{u}_1^{(0)}, \end{aligned} \quad (4.13)$$

where

$$\varepsilon(\mathbf{x}, \zeta) = \begin{cases} 1, & \text{vacuum} \\ 1 + \chi_{star}(\mathbf{x}), & \text{nonabsorptive regions} \\ 1 + \hat{\chi}(\mathbf{x}, \zeta c_0), & \text{absorptive regions.} \end{cases} \quad (4.14)$$

We now specialize to the layered case: $\varepsilon_1 = \varepsilon_1(x_3)$, $\chi = \chi(x_3, t)$. Then, setting $\mathbf{k}^\perp = \boldsymbol{\kappa}$,

$$\begin{aligned} \varepsilon_1^{-1/2} \mathbf{u}_{1\mathbf{k}j}(\mathbf{x}) &= \exp(i\boldsymbol{\kappa} \cdot \mathbf{x}) \boldsymbol{\varphi}_{\mathbf{k}j}(x_3), \\ \varepsilon_0^{-1/2} \mathbf{u}_{1\mathbf{k}j}^{(0)}(\mathbf{x}) &= \exp(i\boldsymbol{\kappa} \cdot \mathbf{x}) \boldsymbol{\varphi}_{\mathbf{k}j}^{(0)}(x_3), \\ \boldsymbol{\varphi}_{\mathbf{k}j}^{(0)}(x_3) &= (2\pi)^{-3/2} \varepsilon_0^{-1/2} \exp(ik_3 x_3), \end{aligned} \quad (4.15)$$

where $\boldsymbol{\varphi}$ is the same function as we encountered before. Again dropping subscripts we obtain

$$\begin{aligned} \boldsymbol{\varphi} &= [\zeta^2 \varepsilon(\zeta) - \mathbf{H}_0(\boldsymbol{\kappa})]^{-1} \left(\frac{\varepsilon_1}{\varepsilon_0} \right)^{1/2} [\zeta^2 - \mathbf{H}_0(\boldsymbol{\kappa})] \boldsymbol{\varphi}^{(0)} \\ &= [\zeta^2 \varepsilon(\zeta) - \mathbf{H}_0(\boldsymbol{\kappa})]^{-1} [\zeta^2 - \mathbf{H}_0(\boldsymbol{\kappa})] \left\{ 1 + [\zeta^2 \right. \\ &\quad \left. - \mathbf{H}_0(\boldsymbol{\kappa})]^{-1} \left[\left(\frac{\varepsilon_1}{\varepsilon_0} \right)^{1/2} - 1 \right] [\zeta^2 - \mathbf{H}_0(\boldsymbol{\kappa})] \right\} \boldsymbol{\varphi}^{(0)}, \end{aligned}$$

with

$$\begin{aligned} \mathbf{H}_0(\boldsymbol{\kappa}) &= \mathbf{H}_0(\boldsymbol{\kappa}, p_3) \\ &= (\boldsymbol{\kappa}^2 + p_3^2) \mathbf{U} - (\boldsymbol{\kappa} + p_3 \mathbf{e}_3)(\boldsymbol{\kappa} + p_3 \mathbf{e}_3) \\ &= (\boldsymbol{\kappa}^2 + p_3^2) \boldsymbol{\Delta}(\boldsymbol{\kappa}, p_3). \end{aligned} \quad (4.16)$$

The term

$$\begin{aligned} &[\zeta^2 - \mathbf{H}_0(\boldsymbol{\kappa})]^{-1} \left[\left(\frac{\varepsilon_1}{\varepsilon_0} \right)^{1/2} - 1 \right] [\zeta^2 - \mathbf{H}_0(\boldsymbol{\kappa})] \boldsymbol{\varphi}^{(0)} \\ &= \lim_{\delta \rightarrow 0} [k^2 - i\delta - \mathbf{H}_0(\boldsymbol{\kappa})]^{-1} \left[\left(\frac{\varepsilon_1}{\varepsilon_0} \right)^{1/2} - 1 \right] (-i\delta) \boldsymbol{\varphi}^{(0)} \end{aligned}$$

vanishes. This follows from the fact that $(\varepsilon_1/\varepsilon_0)^{1/2} - 1$ vanishes for x_3 outside the medium and that the Green's function associated with $[\zeta^2 - \mathbf{H}_0(\boldsymbol{\kappa})]^{-1}$ is given by

$$\begin{aligned} \langle x_3 | [\zeta^2 - \mathbf{H}_0(\boldsymbol{\kappa})]^{-1} | y_3 \rangle &= k^{-2} \delta(x_3 - y_3) \\ &+ \frac{i}{2|k_3|} \boldsymbol{\Delta}(\boldsymbol{\kappa}, -i\partial_{x_3}) \\ &\times \exp(-i|k_3||x_3 - y_3|). \end{aligned} \quad (4.17)$$

Hence

$$\begin{aligned} \boldsymbol{\varphi}_{kj} &= [\zeta^2 \boldsymbol{\varepsilon}(\zeta) - \mathbf{H}_0(\boldsymbol{\kappa})]^{-1} [\zeta^2 - \mathbf{H}_0(\boldsymbol{\kappa})] \boldsymbol{\varphi}_{kj}^{(0)} \\ &= \boldsymbol{\varphi}_{kj}^{(0)} + [\zeta^2 - \mathbf{H}_0(\boldsymbol{\kappa})]^{-1} \mathbf{T}(\zeta) \boldsymbol{\varphi}_{kj}^{(0)}, \end{aligned} \quad (4.18)$$

where the T matrix $\mathbf{T}(\zeta)$ is given by

$$\begin{aligned} \mathbf{T}(\zeta) &= \mathbf{V}(\zeta) + \mathbf{V}(\zeta) [\zeta^2 \boldsymbol{\varepsilon}(\zeta) - \mathbf{H}_0(\boldsymbol{\kappa})]^{-1} \mathbf{V}(\zeta), \\ \mathbf{V}(\zeta) &= \zeta^2 [1 - \boldsymbol{\varepsilon}(\zeta)], \end{aligned} \quad (4.19)$$

and Eq. (4.11) now takes the form

$$[k^2 \boldsymbol{\varepsilon}(-k) - \mathbf{H}_0(\boldsymbol{\kappa})] \boldsymbol{\varphi}_{kj} = 0. \quad (4.20)$$

C. The transfer matrix method

In dealing with one-dimensional second order differential equations with piecewise constant coefficients such as we

encounter in the layered case, it is convenient to use the transfer matrix formalism. It supplies the solutions of Eq. (4.20) up to normalization constants. The latter are then fixed by the behavior of $\boldsymbol{\varphi}$ for large x_3 which follows from Eqs. (4.17) and (4.18). Let us choose a coordinate frame such that the origin is inside the medium and $\mathcal{J} = [x_-, x_+]$ is the interval consisting of x_3 inside the medium and \mathbf{k} in the X_2 - X_3 plane, so $\boldsymbol{\kappa} = \kappa \mathbf{e}_2$ and $\mathbf{n}_1 = \mathbf{e}_1$. In the following $x = x_3$, $p = -i\partial_x$, and $\varepsilon = \varepsilon(x, \zeta)$. At this point we note that $\mathbf{e}_1 \cdot \mathbf{H}_0(\boldsymbol{\kappa}) = (\kappa^2 + p^2) \mathbf{e}_1$, leading to $\mathbf{e}_1 \cdot [\zeta^2 \boldsymbol{\varepsilon} - \mathbf{H}_0(\boldsymbol{\kappa})]^{-1} = [\zeta^2 \boldsymbol{\varepsilon} - \kappa^2 - p^2] \mathbf{e}_1$ and Eqs. (4.18) and (4.20) reduce to the scalar equations $[\varphi_1^{(0)} = \mathbf{e}_1 \cdot \boldsymbol{\varphi}_{k1}^{(0)} = (2\pi)^{-3/2} \exp(ik_3 x_3)]$

$$\varphi_1 = \varphi_1^{(0)} + (k_3^2 - i\delta - p^2)^{-1} \mathbf{t}(\zeta) \varphi_1^{(0)}, \quad (4.21)$$

$$\mathbf{t}(\zeta) = \mathbf{V}(\zeta) + \mathbf{V}(\zeta) (k_3^2 - i\delta - p^2)^{-1} \mathbf{V}(\zeta),$$

and

$$[k^2 \boldsymbol{\varepsilon} - \kappa^2 - p^2] \varphi_1 = 0. \quad (4.22)$$

Thus $\boldsymbol{\varphi}_{k1}$ corresponds with φ_1 and $\boldsymbol{\varphi}_{k2}$ with the set $\{\varphi_j = \mathbf{e}_j \cdot \boldsymbol{\varphi}_{k2}, j=2,3\}$. With $\mu = |k_3|$ and noting that $\mathbf{V}(x_3, \zeta)$ vanishes outside \mathcal{J} , we have

$$\begin{aligned} \boldsymbol{\varphi}(x_3) &= \boldsymbol{\varphi}^{(0)}(x_3) + \frac{i}{2\mu} \boldsymbol{\Delta}(\boldsymbol{\kappa}, -i\partial_{x_3}) \int_{\mathcal{J}} dy_3 \exp(-i\mu|x_3 - y_3|) \langle y_3 | \mathbf{T} | \boldsymbol{\varphi}^{(0)} \rangle \\ &= \begin{cases} \boldsymbol{\varphi}^{(0)}(x_3) + \frac{i}{2\mu} \boldsymbol{\Delta}(\boldsymbol{\kappa}, -\mu) \int_{\mathcal{J}} dy_3 \exp[-i\mu(x_3 - y_3)] \langle y_3 | \mathbf{T} | \boldsymbol{\varphi}^{(0)} \rangle, & x_3 > x_+ \\ \boldsymbol{\varphi}^{(0)}(x_3) + \frac{i}{2\mu} \boldsymbol{\Delta}(\boldsymbol{\kappa}, \mu) \int_{\mathcal{J}} dy_3 \exp[i\mu(x_3 - y_3)] \langle y_3 | \mathbf{T} | \boldsymbol{\varphi}^{(0)} \rangle, & x_3 < x_- \end{cases} \\ &= \begin{cases} \boldsymbol{\varphi}^{(0)}(x_3) + \frac{i}{\mu} \sqrt{\frac{\pi}{2}} \boldsymbol{\Delta}(\boldsymbol{\kappa}, -\mu) \exp(-i\mu x_3) \langle \mu | \mathbf{T} | \boldsymbol{\varphi}^{(0)} \rangle, & x_3 > x_+ \\ \boldsymbol{\varphi}^{(0)}(x_3) + \frac{i}{\mu} \sqrt{\frac{\pi}{2}} \boldsymbol{\Delta}(\boldsymbol{\kappa}, \mu) \exp(i\mu x_3) \langle -\mu | \mathbf{T} | \boldsymbol{\varphi}^{(0)} \rangle, & x_3 < x_- \end{cases} \end{aligned} \quad (4.23)$$

Expanding, $\boldsymbol{\varphi}_{k2} = \sum_{j=2}^3 \varphi_j \mathbf{e}_j$, we obtain upon substitution into Eq. (4.20) and equating coefficients

$$(k^2 \boldsymbol{\varepsilon} - p^2) \varphi_2 + \kappa p \varphi_3 = 0, \quad [k^2 \boldsymbol{\varepsilon} - \kappa^2] \varphi_3 + \kappa p \varphi_2 = 0. \quad (4.24)$$

Hence $p^2 \varphi_2 = -p \kappa^{-1} (k^2 \boldsymbol{\varepsilon} - \kappa^2) \varphi_3$ and substitution into the second equation above results in $p \boldsymbol{\varepsilon} \varphi_3 + \kappa \boldsymbol{\varepsilon} \varphi_2 = 0$, so we end up with a coupled set of first order equations for $\varphi_{2,3}$, which can be written as

$$\partial_x \mathbf{f}(x) = -i \mathbf{A}(x) \mathbf{f}(x), \quad \mathbf{f}(x) = \begin{pmatrix} \varphi_2(x) \\ \boldsymbol{\varepsilon} \varphi_3(x) \end{pmatrix},$$

$$\mathbf{A}(x) = \begin{pmatrix} 0 & \kappa^{-1} (k^2 \boldsymbol{\varepsilon} - \kappa^2) \\ \kappa \boldsymbol{\varepsilon} & 0 \end{pmatrix}, \quad (4.25)$$

with the solution

$$\mathbf{f}(x) = \mathbf{U}(x, x') \mathbf{f}(x'), \quad (4.26)$$

and where the *transfer matrix* $\mathbf{U}(x, x')$ satisfies

$$\partial_x \mathbf{U}(x, x') = -i \mathbf{A}(x) \mathbf{U}(x, x'), \quad (4.27)$$

$$\mathbf{U}(x, x) = 1, \quad \mathbf{U}(x, x'') \mathbf{U}(x'', x') = \mathbf{U}(x, x').$$

Note that even if $\mathbf{A}(x)$ has discontinuities, $\mathbf{U}(x, x')$ is continuous in both arguments. In the layered case, where $\varepsilon(x, \zeta)$ is constant over each layer, we have, for x_0, x_1 , and x_2 in the same layer,

$$\mathbf{U}(x_1, x_2) = \exp[-i\mathbf{A}(x_0)(x_1 - x_2)]. \quad (4.28)$$

Now, for $x > x_+ > x_- > x'$,

$$\mathbf{f}(x) = \mathbf{U}(x, x_+) \mathbf{U}(x_+, x_-) \mathbf{U}(x_-, x') \mathbf{f}(x'), \quad (4.29)$$

where $\mathbf{U}(x_+, x_-)$ consists of the product of a finite number of known matrices, each pertaining to a given layer. Next we note that outside the medium, \mathbf{f} satisfies the free equations of motion

$$\mathbf{f}(x) = \begin{cases} \exp[-i\mathbf{A}_0(x - x_+)] \mathbf{f}(x_+), & x > x_+, \\ \exp[-i\mathbf{A}_0(x - x_-)] \mathbf{f}(x_-), & x < x_-, \end{cases} \quad (4.30)$$

$$\mathbf{A}_0 = \begin{pmatrix} 0 & \kappa^{-1} k_3^2 \\ \kappa & 0 \end{pmatrix},$$

but also has the asymptotic behavior given by Eq. (4.20). This fixes $\mathbf{f}(x_{\pm})$ according to

$$\begin{aligned} \mathbf{f}(x_-) &= (2\pi)^{-3/2} \exp(ik_3 x_-) \\ &\quad \times \sin 2\vartheta [\mathbf{e}_{\mathbf{k}'} \cdot \mathbf{U}(x_+, x_-) \cdot \mathbf{n}_2]^{-1} \mathbf{n}_2, \end{aligned} \quad (4.31)$$

$$\begin{aligned} \mathbf{f}(x_+) &= (2\pi)^{-3/2} \exp(ik_3 x_+) \\ &\quad \times \sin 2\vartheta [\mathbf{e}_{\mathbf{k}'} \cdot \mathbf{U}(x_+, x_-) \cdot \mathbf{n}_2]^{-1} \mathbf{U}(x_+, x_-) \cdot \mathbf{n}_2, \end{aligned}$$

for $k_3 > 0$, whereas for $k_3 < 0$

$$\begin{aligned} \mathbf{f}(x_+) &= (2\pi)^{-3/2} \exp(ik_3 x_-) \\ &\quad \times \sin 2\vartheta [\mathbf{e}_{\mathbf{k}'} \cdot \mathbf{U}(x_+, x_-) \cdot \mathbf{n}_2]^{-1} \mathbf{n}_2, \end{aligned} \quad (4.32)$$

$$\begin{aligned} \mathbf{f}(x_-) &= (2\pi)^{-3/2} \exp(ik_3 x_-) \\ &\quad \times \sin 2\vartheta [\mathbf{e}_{\mathbf{k}'} \cdot \mathbf{U}(x_+, x_-) \cdot \mathbf{n}_2]^{-1} \mathbf{U}(x_+, x_-) \cdot \mathbf{n}_2. \end{aligned}$$

Here ϑ is the angle between \mathbf{k} and the positive X_3 axis, so $\mathbf{e}_{\mathbf{k}} = (0, \sin \vartheta, \cos \vartheta)$ and $\mathbf{n}_2 = (0, \cos \vartheta, -\sin \vartheta)$, whereas $\mathbf{e}_{\mathbf{k}'} = (0, \sin \vartheta, -\cos \vartheta)$. Now $\mathbf{f}(x)$ and hence $\varphi_{\mathbf{k}2}(x)$ can be obtained for any other x from Eq. (4.24).

For φ_1 we have

$$\begin{aligned} \partial_x \mathbf{g}(x) &= -i\mathbf{B}(x) \mathbf{g}(x), \quad \mathbf{g}(x) = \begin{pmatrix} \varphi_1(x) \\ -i\partial_x \varphi_2(x) \end{pmatrix}, \\ \mathbf{B}(x) &= \begin{pmatrix} 0 & -1 \\ \kappa^2 - k^2 \varepsilon & 0 \end{pmatrix}, \end{aligned} \quad (4.33)$$

so

$$\mathbf{g}(x) = \mathbf{V}(x, x') \mathbf{g}(x'), \quad (4.34)$$

where $\mathbf{V}(x, x')$ satisfies Eq. (4.25) with $\mathbf{A}(x)$ replaced by $\mathbf{B}(x)$ and with corresponding further developments, resulting in

$$\begin{aligned} \mathbf{g}(x_-) &= (2\pi)^{-3/2} k_3 \exp(ik_3 x_+) [\mathbf{a} \cdot \mathbf{V}(x_+, x_-) \cdot \mathbf{b}]^{-1} \mathbf{b}, \\ \mathbf{g}(x_+) &= (2\pi)^{-3/2} k_3 \exp(ik_3 x_+) \end{aligned} \quad (4.35)$$

$$\times [\mathbf{a} \cdot \mathbf{V}(x_+, x_-) \cdot \mathbf{b}]^{-1} \mathbf{V}(x_+, x_-) \cdot \mathbf{b}$$

for $k_3 > 0$, whereas for $k_3 < 0$

$$\begin{aligned} \mathbf{g}(x_+) &= (2\pi)^{-3/2} k_3 \exp(ik_3 x_-) [\mathbf{a} \cdot \mathbf{V}(x_-, x_+) \cdot \mathbf{b}]^{-1} \mathbf{b}, \\ \mathbf{g}(x_-) &= (2\pi)^{-3/2} k_3 \exp(ik_3 x_-) \end{aligned} \quad (4.36)$$

$$\times [\mathbf{a} \cdot \mathbf{V}(x_-, x_+) \cdot \mathbf{b}]^{-1} \mathbf{V}(x_-, x_+) \cdot \mathbf{b}.$$

Here \mathbf{a} and \mathbf{b} are given by

$$\mathbf{a} = \begin{pmatrix} k_3 \\ 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ k_3 \end{pmatrix}. \quad (4.37)$$

In actual situations the dielectrics consist of two types of layers which alternate, so layers 1, 3, 5, . . . , $2N-1$ have the same structure, as have layers 2, 4, 6, . . . , $2N$. Then $\mathbf{U}(x_+, x_-)$ and $\mathbf{V}(x_+, x_-)$ become powers of two-layer matrices, whereas an additional single layer matrix is needed if the total number of layers is odd. If the permeability is changing gradually, $\mathbf{U}(x, x')$ and $\mathbf{V}(x, x')$ can still be obtained by means of a direct numerical integration of Eq. (4.25).

V. DISCUSSION

In this section only situations involving normally incident electrons will be considered. Then $\mathbf{v} = v \mathbf{e}_3$ in Eq. (2.32) which now becomes

$$\mathfrak{E}_{\mathbf{k}j} = \frac{\pi e^2}{v^2} |(-kc_0 \mathbf{e}_3 | \varphi_{\mathbf{k}j})|^2, \quad (5.1)$$

$\mathfrak{E}_{\mathbf{k}j} d\mathbf{k}$ being the radiated energy per electron with polarization j and wave number between \mathbf{k} and $\mathbf{k} + d\mathbf{k}$.

A. Comparison with results obtained by the coherent summation procedure

For brevity we refer to the coherent summation method as CSM and to the present approach as STM (scattering theoretical method). The validity of CSM involves the following: (a) The electron velocity is constant. (b) There is no generation of CR. (c) A stack consisting of an odd number of alternating layers, 1, 2, . . . , $2N-1$ is considered. Layers 0 and $2N$ refer to the vacuum at both sides of the stack. (d) The radiation generated at the internal interfaces $i, i+1$ and $i+1, i+2$ has equal amplitude and opposite phase. (e) The radiation generated at the external interfaces 0, 1 and $2N-1, 2N$ is taken equal to that of 2, 3 and 1, 2, respectively. (f) The radiation emerging from the last interface is obtained by coherently summing up the contributions from the individual

interfaces. (g) The dielectric is nonabsorptive. (h) A decrease in the electron beam intensity inside the medium due to scattering can be discounted.

Assumptions (d) and (f) require $|\varepsilon_{i+1} - \varepsilon_i|$ to be small. If not, reflection cannot be neglected and the summation procedure is not justified. Condition (e) is acceptable if the number of layers is sufficiently large. (But then the contribution from the boundary interfaces can be put equal to zero as well.) As mentioned earlier, restrictions (g) and (h) have been removed to some extent by Kaplan *et al.* [2]. If we compare the above list with our results in Secs. II and III, we see that the STM only requires (h), the tradeoff being that now the Helmholtz eigenfunctions must be calculated, which is a straightforward matter. However, if (a)–(g) are met, then the results of both approaches are comparable. To verify this we consider two multilayer structures, *A* and *B*, with incident electron beams of moderate and high energy, respectively. Note that multilayer structures can be expected to boost TR yields at a fixed wavelength only if they are optimized, i.e., the layer thicknesses are such that constructive interference of radiation, produced at subsequent interfaces, takes place. This involves both the electron speed and the radiation velocity in a layer. Structure *A* consists of 207 alternating layers of barium and beryllium, each layer having a width of 66.45 nm. Barium has an absorption edge at a wavelength of 1.59 nm which causes structure *A* to produce intense TR and CR in the vicinity of this wavelength, with an electron beam of sufficient energy (see [2]). We analyze the radiation emitted from structure *A* at this wavelength, for which $\varepsilon_{\text{Ba}} = 1.0013 + 0.00021i$ and $\varepsilon_{\text{Be}} = 0.9988 + 0.000059i$ ($|\varepsilon_{\text{Ba}} - \varepsilon_{\text{Be}}| = 0.0024$). Structure *B* consists of 101 alternating layers of silicon and molybdenum, each with thickness 65 nm. For structure *B* we focus on the wavelength 12.44 nm, for which silicon has an absorption edge, $\varepsilon_{\text{Si}} = 1.043 + 0.0036i$ and $\varepsilon_{\text{Mo}} = 0.88 + 0.0099i$ ($|\varepsilon_{\text{Si}} - \varepsilon_{\text{Mo}}| = 0.16$).

Structure *A* has a small $|\varepsilon_{i-1} - \varepsilon_i|$ and is therefore expected to be accurately described by the CSM, provided there is no emission of CR. For an electron beam of 4.5 MeV, for which no CR is generated, the CSM results agree very well with the corresponding STM ones. Both methods predict production of TR in a cone with axis along X_3 with a maximum at the angle $\theta = \theta_{\text{max}} = 6.66^\circ$ and a narrow distribution around this value. Disregarding photon absorption, the CSM gives a 1% higher intensity than the STM at $\theta = \theta_{\text{max}}$, which number increases to 4.3% if absorption is taken into account (here the augmented CSM, put forward in [2], was employed). This is not too bad in view of the fact that absorption reduces the total photon yield by a factor of 14. When structure *A* is exposed to a more energetic electron beam of 45 MeV, then the TR yield will be supplemented by CR emission generated in the barium layers. This emission of CR is not correctly described by the CSM. In Fig. 1 the predicted radiation yields for the CSM and STM are plotted against the emission angle θ , taking photon absorption into account. We see that the CSM and STM show the same peak at $\theta = 8.85^\circ$ but the CSM fails to describe the yield in the $0^\circ < \theta < 3^\circ$ range correctly. This is to be expected since CR is emitted at $\theta = 1.93^\circ$ which distorts the CSM results in the $0^\circ < \theta < 3^\circ$ range but leaves the peak at $\theta = 8.85^\circ$ unaltered. Without photon absorption the CSM even predicts an infinite yield at $\theta = 1.93^\circ$.

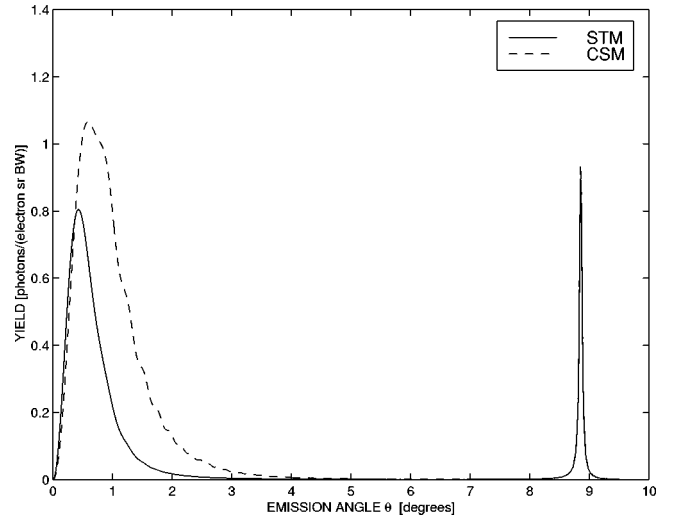


FIG. 1. Combined TR and CR yield at wavelength 1.59 nm in photons/bandwidth sr electron versus emission angle θ for 207 alternating layers of barium and beryllium, each 66.45 nm thick, being bombarded by 45 MeV normally incident electrons.

Since structure *B* has a large $|\varepsilon_{i-1} - \varepsilon_i|$ it is expected that the CSM might fail in describing it, even without CR emission. This turns out to be only partially correct; even for this structure with large $|\varepsilon_{i-1} - \varepsilon_i|$ the CSM appears to be reasonably capable of describing the radiation yield as long as CR is absent. In Fig. 2, where an electron beam of 1.5 MeV is employed (which does not generate CR), the TR yields predicted by the CSM and STM are plotted against emission angle θ . Photon absorption has been included. We see that the CSM and STM agree qualitatively, but there are some noteworthy differences, especially in the minor peaks. For an electron beam of 15 MeV a more pronounced discrepancy appears, as can be seen in Fig. 3. The STM predicts emission in the $0^\circ < \theta < 10^\circ$ range, while the CSM does not. The cause of the discrepancy is not so much the generation of CR that takes place for 15 MeV electrons, but rather the fact that

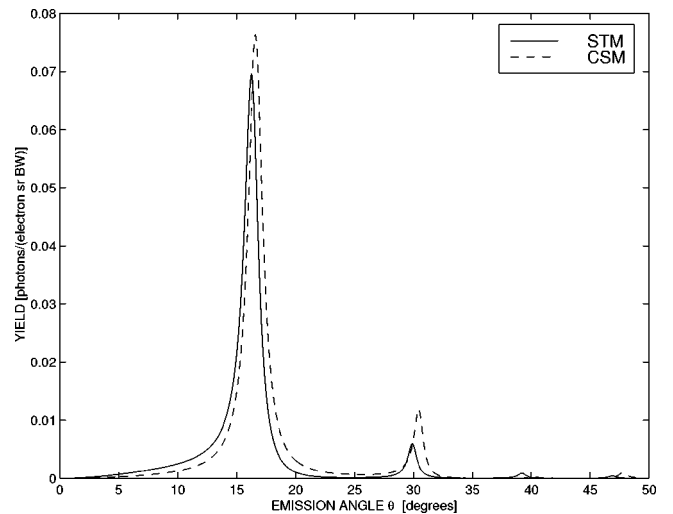


FIG. 2. TR yield at wavelength 12.44 nm in photons/BW sr electron versus emission angle θ for 101 alternating layers of silicon and molybdenum, each 65 nm thick, being bombarded by 1.5 MeV normally incident electrons.

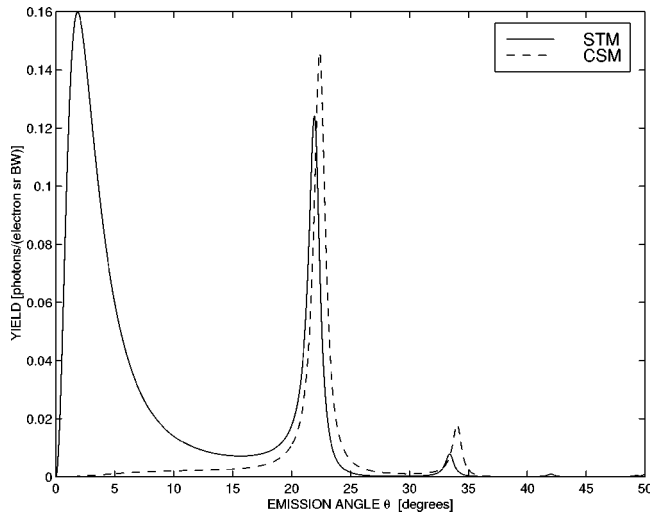


FIG. 3. Combined TR and CR yield at wavelength 12.44 nm in photons/BW sr electron versus emission angle θ for 101 alternating layers of silicon and molybdenum, each 65 nm thick, being bombarded by 15 MeV normally incident electrons.

structure B is optimized for a 1.5 MeV beam. In fact, the CSM turns out to be particularly good in describing peaks in the radiation distribution that arise from positive interference, as in an optimized structure. The peak at $\theta = 1.89^\circ$ that CSM fails to predict corresponds to the TR emitted at the last interface alone; it does not fulfill a coherence condition.

The above examples show that the CSM suffices for describing TR generated in an optimized multilayer structure that has a moderate $|\varepsilon_{i-1} - \varepsilon_i|$, provided there is no generation of CR. If $|\varepsilon_{i-1} - \varepsilon_i|$ is large the CSM can be slightly inaccurate, as seen in Fig. 2. If CR emission takes place or the multilayer structure is not optimized then more serious discrepancies are found, as seen in Figs. 1 and 3. Finally, we give a comparison of the CSM and quantum STM predictions for TR produced by a single slab in Fig. 4. It is seen that the quantum STM result correctly becomes zero for

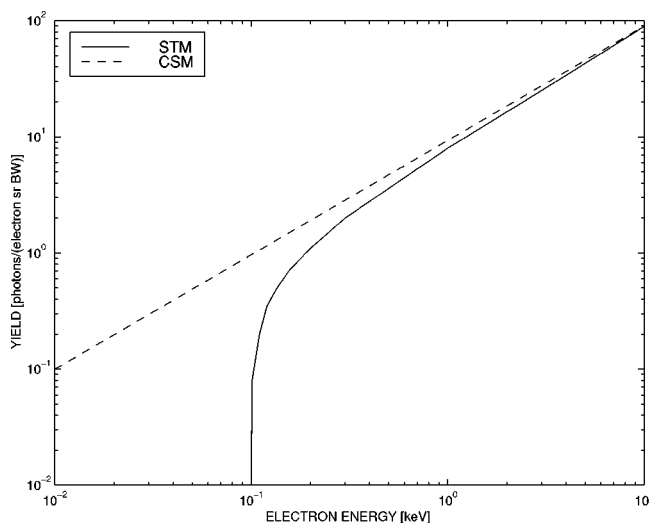


FIG. 4. TR yield at wavelength 12.44 nm and at optimized emission angle in photons/BW sr electron versus electron energy for a single 10 μm thick silicon slab being bombarded by normally incident electrons.

electron energies smaller than the observed photon energy, whereas the CSM yield gradually decreases with decreasing electron energy.

In a separate publication [13], we shall present detailed numerical results about TR and CR yields for a variety of situations.

B. Further theoretical considerations

The approach we have followed gives a considerable improvement over the coherent summation procedure and is likely to be optimal within the framework of the linear phenomenological Maxwell's equations (ME). The latter can be looked upon as a type of mean field approximation to the underlying microscopic physics. As such it does not include processes such as electron scattering from the individual particles (atoms or ions) constituting the material, and accordingly the associated bremsstrahlung is also missing. Attenuation of the electron beam can be included in much the same way as has been done in the CSM, as we have done in some of our calculations [12]. If we use the formalism of Sec. II, where the current is prescribed, we can try to include electron scattering by specifying the path of the electron differently. In fact, an individual electron will perform a lossy random walk (it can lose energy at each encounter with one of the particles in the material). This makes the current a random quantity and an additional averaging is required to arrive at the final result. Alternatively, a formulation in terms of a transport equation can be contemplated. The basic idea is then to add an additional term V , which accounts for the interaction of the electron with the individual atoms, to the Hamiltonian

$$H = H_{electron} + H_{field} + H_{int} \quad (5.2)$$

that is at the basis of our results presented in Sec. III. In its simplest form, disregarding inelastic processes, such as excitation of the atoms and energy-momentum transfer, V consists of a sum of potentials,

$$V = \sum_j V_j(\mathbf{x} - \mathbf{x}_j), \quad (5.3)$$

where V_j is the potential between the electron and the j th atom or ion in the material, \mathbf{x} being the electron coordinate and \mathbf{x}_j the position of the j th atom. Depending on the type of material, \mathbf{x}_j is a lattice point of a crystal or randomly distributed in an amorphous material. Note that without the coupling with the field we are dealing with the quantum Lorentz gas. The next step would be to make a binary collision expansion of the T matrix associated with V . The leading term in this expansion describes the situation where the individual scattering events are independent (Boltzmann equation level). If this is the case, i.e., higher order terms can be dropped, the resulting expression for radiation production not only accounts for electron beam attenuation but also describes the radiation produced by the deflected electrons (which we expect to be of minor importance). However, this program is quite difficult to implement; either a density operator approach or the Bethe-Salpeter equation is needed, a further complication being the breaking of the translation invariance in the X_1 and X_2 directions. For randomly distributed atoms it can be restored by averaging over the positions \mathbf{x}_j but that can only be justified if the underlying stochastic process is ergodic.

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